Ergodic Capacity of a SIMO System Over Nakagami-q Fading Channel

Md. Sohidul Islam* and Mohammad Rakibul Islam

Dept. of Electrical and Electronic Engineering, Islamic University of Technology (IUT), Gazipur, Bangladesh

*Email : msohidul@iut-dhaka.edu

ABSTRACT

In this paper, we present an analytical expression for ergodic capacity of single-input multiple-output (SIMO) wireless communication systems over Nakagami-q fading channels. We derive analytical expressions both for identically independent distributed (i.i.d.) and non-identically independent distributed subchannels (non-i.i.d). The capacity calculation has been carried out using approximations on zeroth-order modified Bessel functions of the first kind. We present numerical results that have been obtained on small limit argument approximation. We assumed that channel state information (CSI) is available at the receiver, but not at the transmitter. We found that the ergodic capacity of the Nakagami-q fading subchannels is proportional to the transmitted signal to noise ratio of the subchannels.

1. INTRODUCTION

In real life wireless communication systems, we expect to download and upload information faster as possible. To make it faster we need to increase the capacity. Information theory has demonstrated the enormous potential capacity of wireless communication systems with multiple antennas at both transmitter and the receiver, when the channel exhibits rich scattering and channel-state information (CSI) is available at the receiver [1]. The concept of information theoretic capacity was first introduced by Shannon [2] to characterize fundamental limits of communication over wireless channel. In [3] they investigated information outage probability of orthogonal space-time block coded multiple-input multiple-output (MIMO) systems operating over Hoyt distributed fading channels for independent and identically distributed fading channels. They also investigated exact-closed form expressions for non-independent and identically distributed fading channels using saddle point approximation method to approximate information outage probability. Here in this paper we present channel capacity formulas of single-input multiple-output (SIMO) system over Nakagami-q fading channels.

Majority of the research related to SIMO capacity has focused on Rayleigh, Rician and Nakagami-m distributions. There are other types of fading distributions which serves good models under certain circumstances. It allows the modeling [4] of propagation environment without a dominant component over the scattered waves (a situation of Non-line of Sight). It has found application in the error performance evaluation of digital communication systems over generalized fading channel [5]. Also, this fading channel is normally observed in satellite link subject to strong ionospheric scintillation and heavily shadowed environment [6]. Instantaneous signal to noise ratio follows the log-square-Hoyt distribution in inter-satellite communications [7]. Recently, the Nakagami-q model is being used more frequently in performance analysis and other studies related to mobile radio communications.

The rest of the papers organized as follows. Section 2 describes the Nakagami-Hoyt distribution. System model is presented in Section 3. Section 4 shows the probably distribution function (PDF) calculations for both non-identically independent and identically independent Nakagami-q fading channels. The ergodic capacity calculations are described in Section 5. In Section 6 simulation and results are shown. Finally, Section 7 describes the concluding remarks of this work.

2. NAKAGAMI-q DISTRIBUTION

The Nakagami-q distribution is the distribution of modulus of complex Gaussian random variables with zero mean and unequal variances. In [8], the probably distribution function (PDF) of Nakagami-q distribution is given by,

\[ p_f(\gamma) = \frac{(1 + q^2)\gamma \exp(-\frac{(1 + q^2)\gamma}{2q})}{2q^2q} I_0\left(\frac{(1 - q^2)\gamma}{4q^2}\right) \quad \gamma \geq 0 \] (1)

Where \( I_0(.) \) is the zeroth-order modified Bessel function of the first kind and \( q \) is the Nakagami-q fading parameter ranges from 0 to 1. \( \gamma \) and \( \bar{\gamma} \) are average SNR and instantaneous SNR respectively. The Nakagami-q distribution spans the range from one-sided Gaussian fading \( (q=0) \) to Rayleigh fading when \( q=1 \).

3. SYSTEM MODEL

In single-input multiple-output (SIMO) wireless communication system, transmitter is equipped with single transmitting antenna and receiver with multiple receiving antennas \( (N_R) \). Let \( p_t \) being transmitted signal power. Let us consider the received signal is corrupted by additive white noise.
gaussian noise (AWGN) as well as the transmission is carried over non-identically independent Nakagami-q fading subchannels. The received signal [9] vector can be expressed as

$$ \mathbf{r} = \mathbf{h}\mathbf{x} + \mathbf{z} \quad (2) $$

where \( \mathbf{h} \) is a \( 1 \times n_R \) vector defining the subchannel gains from transmitter to each receiver antenna, \( \mathbf{h} = [h_1, h_2, \ldots, h_n] \). \( \mathbf{x} \) is the transmitted signal, and \( \mathbf{z} \) is an \( n_R \times 1 \) vector of an i.i.d. zero mean complex additive white gaussian noise (AWGN).

In SISO wireless communication systems Shannon channel capacity is the maximum mutual information between the transmitted signal and the received one [10], capacity can be expressed as

$$ C = W \log_2 (1 + \rho) \quad (3) $$

Where \( \rho = P / \sigma^2 \) the transmitted signal to noise ratio and \( W \) is the transmission bandwidth. The channel is power limited in the sense that \( P = E[|x|^2] \). In case of multiantenna communication systems (SIMO, multiple input single output (MISO) and MIMO), the ergodic capacity is defined as [9], [11])

$$ C = E[W \log_2 (1 + \rho^2)] \quad (4) $$

where \( E[.] \) denotes the expectation operator which is evaluated over the pdf of channel gain matrix, \( \mathbf{h} \). Here \( \mathbf{H} \) denotes the complex conjugate transpose. For SIMO [11], \( \mathbf{h} \) is a vector of size \( 1 \times n_R \), then

$$ \mathbf{h}^H\mathbf{h} = \sum_{i=1}^{n_R} |h_i|^2 \quad (5) $$

The ergodic capacity becomes

$$ C = E[W \log_2 (1 + \rho \sum_{i=1}^{n_R} |h_i|^2)] \quad (6) $$

Using small argument limit approximation [12], the zeroth-order modified Bessel function of the first kind can be approximated as \( I_0 \approx 1 \)

According to above approximation the Nakagami-q distribution given in (1) becomes

$$ p_q(\gamma) = \frac{(1 + q^2)^\gamma e^{-(1+q^2)\gamma}}{2q^\gamma}, \gamma > 0 \quad (7) $$

Equation (7) is Nakagami-q distribution under small limit argument approximation. If we set fading parameter \( q=1 \) in equation (7) then it becomes Rayleigh fading distribution.

Now we will derive the expression for Nakagami-q distribution using large argument approximation. According to [13], the zeroth order modified Bessel functions can be written as

$$ I_0(\ell) \approx \frac{e^\ell}{\sqrt{2\pi \ell r^2}} \quad (7) $$

where \( rt >> 1 \)

Under this approximation the zeroth order modified Bessel function of Nakagami-q distribution (1) becomes

$$ I_0\left(\frac{(1 - q^2)\gamma}{4q^2\gamma}\right) \approx \frac{e^{-\frac{(1-q^2)\gamma}{4q^2\gamma}}}{2\pi \frac{(1-q^4)\gamma}{4q^2\gamma}} \quad (9) $$

Using the above approximation Nakagami-q distribution given (1) becomes

$$ p_q(\gamma) = \frac{(1 + q^2)^\gamma e^{-(1+q^2)\gamma}}{2q^\gamma}, \beta = \frac{(1 + q^2)^2}{4q^2\gamma} \quad (10) $$

Equation (10) is simplified as follows

$$ p_q(\gamma) = \psi e^{-\beta \gamma} q^\beta \gamma \quad (11) $$

Where \( \psi = \frac{(1+q^2)^2}{2q^\gamma}, \beta = \frac{(1 + q^2)^2}{4q^2\gamma} \) and \( \eta = \frac{(1 - q^4)}{4q^2\gamma} \quad (12) $$

Equation (11) is Nakagami-q distribution under large limit argument approximation.

**4. PROBABILITY DISTRIBUTION FUNCTION CALCULATION**

The following PDF calculation is shown for small limit argument approximation. According to [9], the ergodic capacity of single-input multiple-output (SIMO) systems requires the exact knowledge of the PDF of the subchannel gains. In (6), the PDF of envelope of \( h_i \) denote by \( \gamma_i \) in Nakagami-q distribution (1). Now define

$$ y = \sum_{i=1}^{n_R} |h_i|^2 \quad (13) $$

We have \( y \sim \chi_{2n_R}^2 \), where \( \chi_{2n_R}^2 \) is a central chi-square variable with \( 2n_R \) degrees of freedom.

The distribution of \( y \) considered under the following two cases:

**4.1 Case A (Non-identically independent Nakagami-q)**

In this case, the distribution [14] of \( y \) is approximated by:

$$ p(y) \equiv \frac{\kappa_n y^{M_0-1} e^{-\lambda y}}{(M_0-1)!}, \quad (y \geq 0) \quad (14) $$

where

$$ M_0 = \left( \sum_{i=1}^{n_R} \Omega_i \right)^2, \kappa_n = \frac{(1 + M_0^2)^2}{2M_0}, \lambda = \frac{(1 + M_0^2)^2}{2M_0^2\Omega_T} \quad (15) $$
\( q_i \) and \( \Omega \) are the \( i \)th subchannels fading parameter and signal to noise ratio respectively.

### 4.2 Case B (Identically independent Nakagami-q)

For this case all subchannels have identical gains and \( q_i = q \), \( \Omega = \Omega \) and \( M_i = q n_k \). The distribution of \( y \) is approximated by:

\[
p(y) \approx \frac{\kappa^{qn_k - 1} e^{-\kappa y}}{(qn_k - 1)!}, \quad (y \geq 0)
\]

(16)

Where \( \kappa = \frac{(1 + (qn_k)^2)^2}{2(qn_k)^2} \) and \( \lambda = \frac{(1 + (qn_k)^2)^2}{4(qn_k)^2} \Omega \)

(17)

### 5. Analytical expression for SIMO channel Ergodic Capacity

The following capacity calculation is shown for small limit argument approximation. In this section we will derive analytical expressions for identically independent as well non-identically independent Nakagami-q fading SIMO channels. The ergodic capacity [9] of the system can be obtained by using (4),

\[
C = \int_0^\infty p(y) \log_2 (1 + \rho y) dy
\]

(18)

Substituting (14) into (18) and the identically independent Nakagami-q fading capacity becomes:

\[
C_{\text{non-id}} = \frac{\kappa_0}{\rho \Omega^2 (M - 1)!} \int_0^\infty y^{M_i - 1} e^{-\frac{y}{\rho}} \ln(1 + y) dy
\]

(19)

For identically independent Nakagami-q fading case, substituting (16) in to (18) the ergodic capacity becomes:

\[
C_{\text{id}} = \frac{\kappa}{\rho \Omega^2 (q n_k - 1)!} \int_0^\infty y^{qn_k - 1} e^{-\frac{y}{\rho}} \ln(1 + y) dy
\]

(20)

The analytical expression [1] for the SIMO channel capacity from Equation (4) is given by following theorem.

**Theorem 1** When complete channel state information (CCSI) is available at the receiver, the SIMO ergodic capacity for Rayleigh fading channel can be expressed as a function of the number of receiver antennas \( M \) and the SNR \( \rho \) at each receive antenna. Its analytical expression is

\[
C_{\text{Erg-SIMO}}(M, \rho) = \frac{1}{\ln 2} \left[ 1 + \sum_{i=1}^{M-1} \frac{1}{(M-i)!} \left( \frac{1}{\rho} \right)^{M-i} \right] \times
\]

\[
\frac{1}{\rho} \left[ E_1 \left( \frac{1}{\rho} \right) \right] + \frac{1}{\ln 2} \sum_{i=1}^{M-1} \sum_{k=1}^{M-i} \left( \frac{1}{\rho} \right)^{M-i-k} \times
\]

\[
\frac{1}{k \times (M-i-k)! \times (k-i)!}
\]

(21)

Where \( k! \) is the factorial of a nonnegative integer \( k \). \( E_1(x) \) is the exponential integral.

Equation (21) is the exact closed-form capacity expression for SIMO systems which avoids numerical integrations or Monte Carlo simulations. In practice, the values of \( \rho \) could be calculated in advance and stored in lookup tables.

**Proof of theorem 1** In PDF calculation as we mentioned earlier that the probability distribution function (PDF) of \( y = \chi^2_2 \) is

\[
p(y) = \frac{1}{(M-1)!} y^{M-1} e^{-y}; \quad y \geq 0
\]

(21a)

Thus,

\[
E_y \{ \log_2 (1 + \rho y) \} = \int_0^\infty p(y) \log_2 (1 + \rho y) dy =
\]

\[
\frac{1}{\rho^M (M-i)!} \int_0^\infty y^{M-i} e^{-\frac{y}{\rho}} \ln(1 + y) dy
\]

(21b)

Where

\[
\int_0^\infty y^{M-i} e^{-\frac{y}{\rho}} \ln(1 + y) dy =
\]

\[
\rho \int_0^\infty y^{M-i} e^{-\frac{y}{\rho}} dy + \rho^2 (M-1) \int_0^\infty y^{M-i} e^{-\frac{y}{\rho}} \ln(1 + y) dy = \cdots =
\]

\[
\sum_{i=1}^{M-1} \frac{(M-1)!}{(M-i)!} \rho^i \int_0^\infty y^{M-i} e^{-\frac{y}{\rho}} dy + \rho^{M-1} (M-1) \int_0^\infty e^{-\frac{y}{\rho}} \ln(1 + y) dy
\]

(21c)

Equation (21c) can be divided in to two parts:

\[
\int_0^\infty y^{M-i} e^{-\frac{y}{\rho}} dy = \int_1^\infty \frac{(y-1)^{M-i}}{y} e^{-\frac{y}{\rho}} e^{-y \rho} dy =
\]

\[
e^{y \rho} \sum_{k=0}^{M-i} C_{M-i-k}^k \int_1^\infty y^{k-1} e^{-\frac{y}{\rho}} dy =
\]

\[
(1)^{M-i} e^{-y \rho} E_1 \left( \frac{1}{\rho} \right) + \sum_{i=1}^{M-i} C_{M-i}^i \sum_{k=0}^{M-i-k} \frac{(k-i)!}{(k-i)!} \times
\]

\[
E_1 \left( \frac{1}{\rho} \right)
\]

(21d)

and

\[
\int_0^\infty e^{-\frac{y}{\rho}} \ln(1 + y) dy = \rho \int_0^\infty y^{M-i} e^{-\frac{y}{\rho}} dy =
\]

\[
\rho e^{y \rho} E_1 \left( \frac{1}{\rho} \right)
\]

(21e)
Equations (21a)-(21e) can be combine to give equation (21).
To evaluate the ergodic capacity given in non-identically independent Nakagami-q fading case (19) and identically independent Nakagami-q fading case (20) we can use following result from equation (21):

\[ E_q(x) = \int_0^\infty e^{x u} du = \frac{(\alpha - 1)!}{\mu^\alpha} \left( \sum_{i=1}^{\infty} \frac{1}{(\alpha - 1)!} (-\mu)^{\alpha - i} \right) e^{\mu E_q(\mu)} \]

\[ + \sum_{k=1}^{\infty} \frac{(-1)^{k+i} E_q(\mu)}{k!(\alpha - i - k)(k - 1)!} \]

where \( E_q(x) \) is the exponent integral defined as,

\[ E_q(x) = \int_0^x e^{\alpha} du \quad (23) \]

By using (21), the ergodic capacity of Non-identically independent Nakagami-q (19) is approximated by:

\[ C_{non-\text{id}} \approx \frac{\kappa}{\rho^\mu (M_0 - 1)! \ln 2} \left[ \frac{(M_0 - 1)!(\lambda_0 / \rho)^{M_0 - 1}}{\left( \frac{\lambda_0}{\rho} \right)^{M_0}} \sum_{i=1}^{\infty} \frac{1}{(M_0 - 1)!} (-\lambda_0 / \rho)^{M_0 - i} \right] e^{\mu E_q(\lambda_0 / \rho)} + \]

\[ \frac{(M_0 - 1)!}{(\lambda_0 / \rho)^{M_0}} \sum_{i=1}^{\infty} \frac{1}{(M_0 - 1)!} (-\lambda_0 / \rho)^{M_0 - i} \left[ \frac{\lambda_0}{\rho} \right]^{M_0 - i} \sum_{k=1}^{\infty} k!(\lambda_0 - i - k)(k - 1)! \] \]

The ergodic capacity for identically independent Nakagami-q fading subchannels (20) is approximated by:

\[ C_{\text{id}} \approx \frac{\kappa}{\rho^\mu (q n_R - 1)! \ln 2} \left[ \frac{(qn_R - 1)!(\lambda / \rho)^{qn_R - 1}}{\left( \frac{\lambda}{\rho} \right)^{qn_R}} \sum_{i=1}^{\infty} \frac{1}{(qn_R - 1)!} (-\lambda / \rho)^{qn_R - i} \right] e^{\mu E_q(\lambda / \rho)} + \]

\[ \frac{(qn_R - 1)!}{(\lambda / \rho)^{qn_R}} \sum_{i=1}^{\infty} \frac{1}{(qn_R - 1)!} (-\lambda / \rho)^{qn_R - i} \left[ \frac{\lambda}{\rho} \right]^{qn_R - i} \sum_{k=1}^{\infty} k!(qn_R - i - k)(k - 1)! \] \]

6. SIMULATION AND RESULTS

In this section, we will investigate the variation of the ergodic channel capacity with respect to various parameters for non-identically independent and identically independent distributed channels. For all of the given figures in this paper ergodic channel capacity has been shown against the transmitted signal to noise ratio and number of receivers.

From Fig.1, for all (case 1, case 2 and case 3) cases it is obviously found that ergodic capacity of non-identically independent Nakagami-q fading channels is increasing along with the increasing transmitted signal to noise ratio.

In Fig.1 case 1 represents non-identically independent Nakagami-q with three subchannels with fading parameters \( q = 0.525, 0.525, 0.525 \) and \( \Omega = 3, 5, 7 \). In this case capacity at 0 dB is 15.81 bits/sec/Hz and increasing capacity at 30 dB is found 18.76 bits/sec/Hz.

In Fig.1 case 2 represents non-identically independent Nakagami-q with three subchannels with fading parameters \( q = 0.25, 0.25, 0.425 \) and \( \Omega = 3, 5, 7 \) and capacity at 30 dB is found 13.35 bits/sec/Hz.

In Fig.1 case 3 represents non-identically independent Nakagami-q with three subchannels with fading parameters \( q = 0.425, 0.425, 0.425 \) and \( \Omega = 1, 3, 5 \). In case 3 ergodic capacity at 0 dB is 10.64 bits/sec/Hz and increasing capacity at 30 dB capacity is 12.40 bits/sec/Hz.

These different cases of simulation show that our derived equation is working perfectly. For each case we found the same result that we expect. That is capacity of the system increases with increasing transmitted signal to noise ratio.
The simulation results present in Fig. 2 shows ergodic capacity of the identically independent distributed. Nakagami-q subchannels with respect to average SNR. In this simulation fading parameter is set to 0.85 and number of receiving antennas, $n_R = 2$. For different values of $\Omega$ ergodic capacity is shown. As the value of $\Omega$ increases the capacity increases along with different values of transmitted SNR. So, average SNR is a function of channel capacity.

Fig. 3 represents ergodic capacity for the identically independent Nakagami-q subchannels with respect to number of receivers for different values of transmitted signal to noise ratios. For this simulation, we used fading parameter 0.25 and $\Omega = 9$. It is observed that as the number of receivers increases for each value of the transmitted SNR the ergodic capacity of the system increases. So, number of receiver is a function of channel capacity.

7. Conclusion

In this paper, we have derived analytical expressions for the ergodic capacity of both i.i.d. and non-identically independent Nakagami-q fading channels using small argument approximation of zeroth-order modified Bessel function of first kind. We present numerical and simulation results for both i.i.d. and non-identically independent Nakagami-q fading single input multiple output channels. It is found that the ergodic capacity of the Nakagami-q fading subchannels is proportional to the transmitted signal to noise ratio of the subchannels.

REFERENCES


